Section Two: Calculator-assumed (80 Marks)

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

Working time: 100 minutes.

Question 9 (5 marks)

For events A and B;

$$P(A) = 0.5, \quad P(B \mid A) = 0.3 \quad \text{and} \quad P(A \cup B) = 0.8.$$

(a) Calculate $P(A \cap B)$ (1 mark)

$P(A \cap B) = P(A) \times P(B \mid A)$

$= 0.5 \times 0.3$

$= 0.15$

(b) Calculate $P(B)$ (1 mark)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \quad 0.8 = 0.5 + P(B) - 0.15$

$\therefore \quad P(B) = 0.45$

(c) Calculate $P(\overline{A} \cap B)$ (1 mark)

$P(\overline{A} \cap B) + P(A \cap B) = P(B)$

$\therefore \quad P(\overline{A} \cap B) + 0.15 = 0.45$

$\therefore \quad P(\overline{A} \cap B) = 0.3$

(d) Are events A and B independent? Justify your answer. (2 marks)

$P(B) = 0.45$

$P(B \mid A) = 0.3$

So $P(B) \neq P(B \mid A)$

$\therefore$ A and B are not independent.

Question 10 **(8 marks)**

Suppose that 9% of people in a certain community live alone.

Sixty percent of people who live alone own pets, whereas only 3% of people who do not live alone own pets.

(a) What is the probability that a person chosen at random from this community owns a pet? (2 marks)

$0.09 \times 0.6 + 0.91 \times 0.03$

$= 0.0813$

(b) What is the probability that a randomly-selected person from this community neither lives alone nor owns a pet? (1 mark)

0.91×0.97

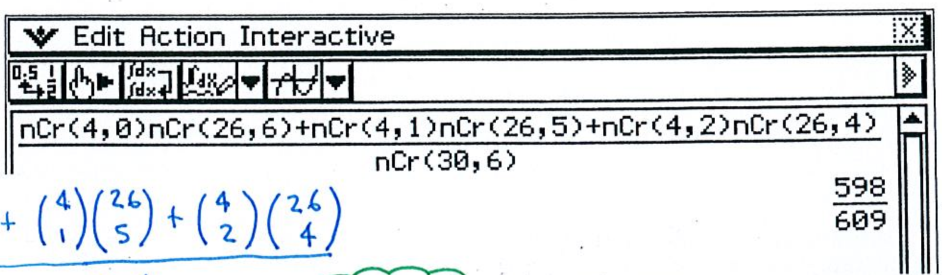
$= 0.8827$

(c) What percentage of people from this community who own a pet live alone? (2 marks)

$$\frac{0.09 \times 0.6}{0.0813}$$

$$= \frac{180}{271} = 66.4\%$$

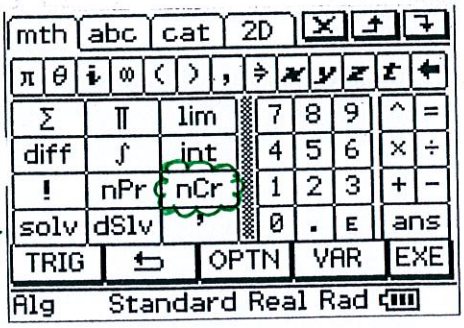
(d) In a group of 30 people in this community, four live alone. If six people are selected from this group, what is the probability that no more than two of them live alone? (3 marks)



$$\frac{\binom{4}{0}\binom{26}{6} + \binom{4}{1}\binom{26}{5} + \binom{4}{2}\binom{26}{4}}{\binom{30}{6}}$$

$$= \frac{598}{609}$$

CP Note: 'nCr' is on the mth-CALC keypad.



See next page

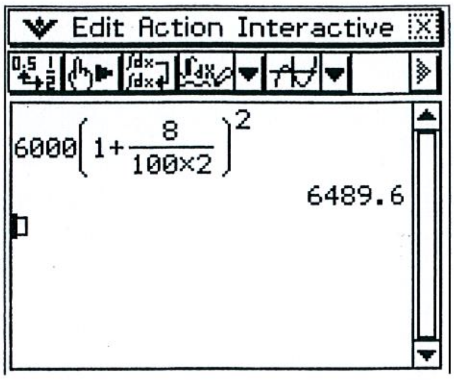
Question 11 **(5 marks)**

When an amount $\$A$ is invested at an interest rate of $r\%$ per annum, compounded n times per year, the value $\$V$ of the investment after one year is given by $V = A\left(1+\frac{r}{100n}\right)^n$.

Kelvin invests \$6000 at 8% per annum interest for one year.

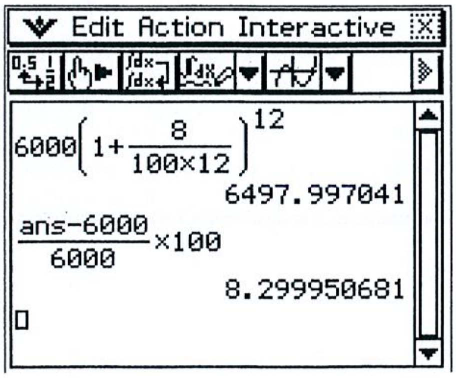
(a) What is the value of the investment at the end of the year if interest is compounded twice per year? (1 mark)

\$6489.60



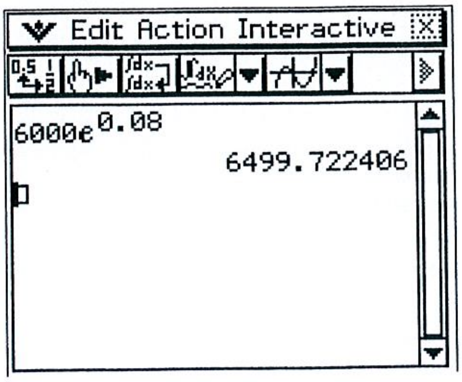
(b) If interest is compounded monthly, by what percentage does the investment increase over the course of the year? (2 marks)

The value becomes
\$6498.00
This is an increase of
8.3%



(c) If the investment were to be compounded more frequently, could the value of Kelvin's investment rise above \$6500 at the end of the year? Justify your answer. (2 marks)

If the investment was compounded continuously it have a value of
\$6499.72
∴ It's value can't exceed \$6500.00 by more frequent compounds.

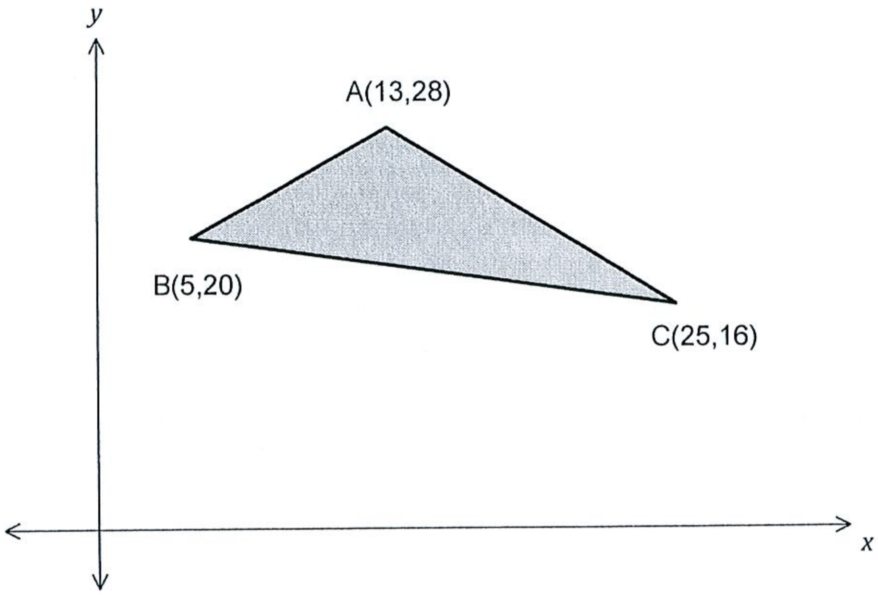


Question 12 **(9 marks)**

Each day, a company produces x thousand units of commodity X and y thousand units of commodity Y.

Each unit of commodity X earns a profit of \$21, and each unit of commodity Y earns a profit of \$15.

The feasible region for the company's daily production schedule is the triangle shown below.



(a) Determine the inequality satisfied by x and y that corresponds to the edge AB of the feasible region. (2 marks)

$m = 8/8 = 1$

$\therefore\ y = x + c$

$\therefore\ 20 = 5 + c \Rightarrow c = 15$

$\therefore$ equation is $y = x + 15$

$\therefore$ inequality is

$y \leq x + 15$

(b) Determine the maximum possible daily profit. (2 marks)

$P = 21x + 15y$ (= profit in '000's)

at A(5,20) $P = 405$

B(13,28) $P = 693$

C(25,16) $P = 765$

$\therefore$ max. profit is \$765,000

(c) The company decides that the amount of commodity Y produced cannot be more than three times the amount of commodity X.

How does this additional constraint affect the maximum possible profit? Justify your answer. (2 marks)

New constraint is $y \leq 3x$

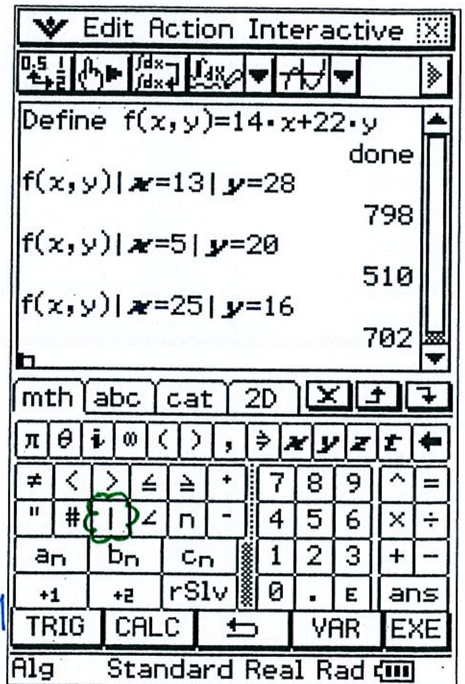
$C(21,16)$ satisfies this so there is no change in max. profit.

(d) Changing market conditions will cause changes to the unit profits for commodities X and Y.

On each of the next seven days, the unit profit for commodity X will fall by \$1, and the unit profit for commodity Y will rise by \$1.

Describe how these changes will affect the maximum possible daily profit over the course of the next week. (3 marks)

Day	Profit at vertex A	B	C
1	708	420	756
2	723	435	747
3	738	450	738
4	753	465	729
5	768	480	720
6	783	495	711
7	798	510	702



The max profit falls by \$9000 per day for the first three days then, by changing to vertex A, the max. profit rises by \$15,000 per day for the rest of the week

CP Note:

- The '|' (given that) symbol is on the mth - OPTN keypad.
- This table is generated by changing the objective function $f(x,y)$ and then values at the vertices are recalculated.

Note:
An alternative solution is provided on page 21.

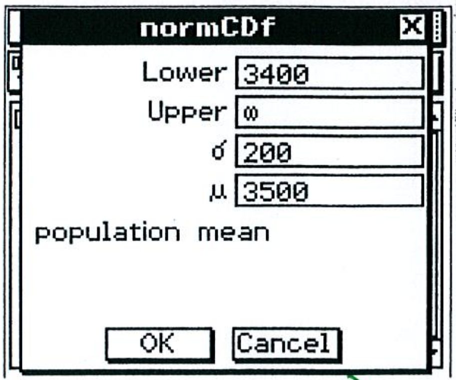
Question 13 **(7 marks)**

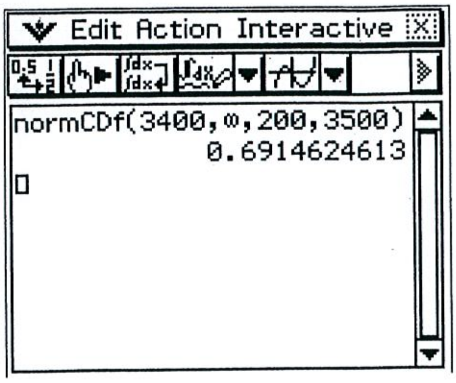
The lifetimes of Glowbrite light bulbs are normally distributed with mean 3500 hours and standard deviation 200 hours.

(a) What is the probability that the lifetime of a randomly-selected bulb is at least 3400 hours? (1 mark)

$P(T > 3400)$

$= 0.6915$



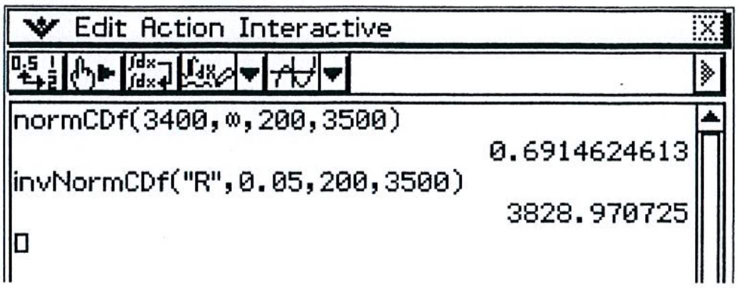


CP Note:
All statistical calculations are done via Interactive Distribution or Inv. Distribution

(b) Calculate t, given that 5% of Glowbrite bulbs last longer than t hours. (1 mark)

$P(T > t) = 0.05$

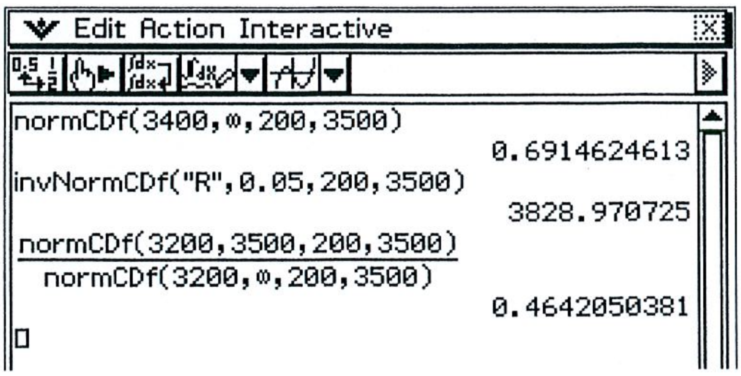
$\therefore t = 3829$



(c) What is the probability that a Glowbrite bulb will last no more than 3500 hours, if it has already lasted 3200 hours? (3 marks)

$$\frac{P(3200 \leq T \leq 3500)}{P(T \geq 3200)}$$

$= 0.4642$



(d) The company also produces Ultrabrite light bulbs, whose lifetimes are also normally distributed, with the same standard deviation of 200 hours but with a possibly different mean μ hours.

A quality control expert at the company wishes to estimate μ using the mean lifetime of a random sample of Ultrabrite bulbs.

How large should the sample be in order to be 95% confident that the estimate will be no more than 10 hours in error? (2 marks)

$$1.96 \times \frac{200}{\sqrt{n}} = 10$$

$\therefore \quad n = 1536.64$

$\therefore$ Sample must be at least 1537

Question 14 **(5 marks)**

During a volcanic eruption a rock is ejected from the top of the volcano. The rock rises upward and then falls onto a flat plain 1500 metres below the top of the volcano. During its flight, the vertical velocity of the rock, v m/s, is given by

$$v = 160 - 9.8t$$

where t seconds is the time after the ejection of the rock.

(a) How high does the rock rise above the top of the volcano? (3 marks)

$h(t) = \int v(t)\,dt = 160t - 4.9t^2 + c$

$h(0) = c = 0$

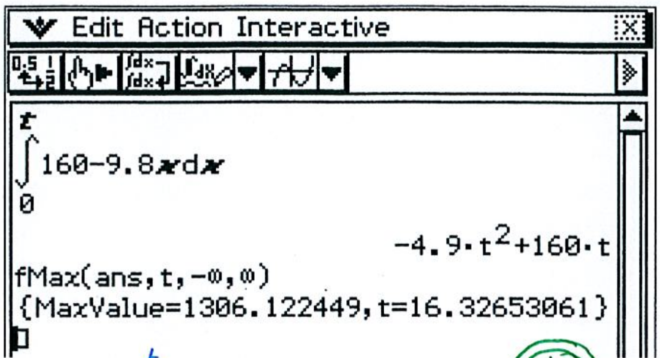
max. height when $t = -\frac{b}{2a} = \frac{-160}{-9.8}$

i.e. after 16.33 secs

$h(16.33) = 160 \times 16.33 - 4.9(16.33)^2$

$= 1306.12$

i.e. max height is 1306.12 m



$h(t) = \int_0^t 160 - 9.8x\,dx$

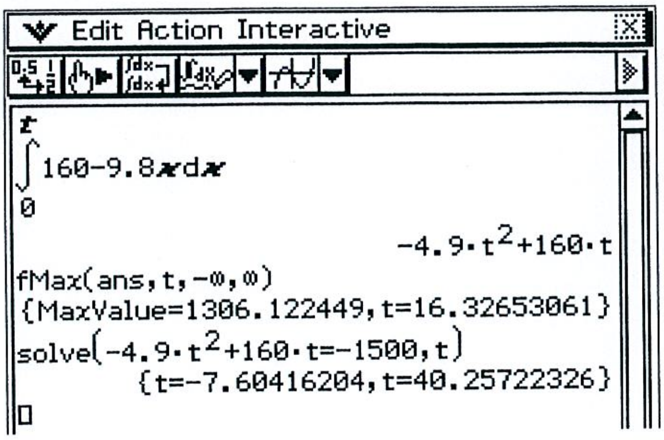
$= -4.9t^2 + 160t$

max height is 1306.12 m after 16.33 secs.

(b) How long does it take for the rock to reach the plain below? (2 marks)

$160t - 4.9t^2 = -1500$

$t = 40.26$ secs $(t>0)$

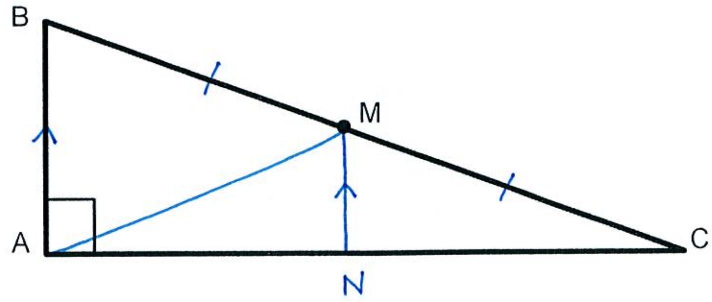


CP Note:

The 'fMax' command finds the maximum value over the chosen domain by considering local maximums and endpoints. In the case of quadratic f^ns this works well, but it may not for other functions.

Question 15 **(4 marks)**

In the diagram below ABC is a right-angled triangle, and M is the mid-point of the hypotenuse BC.



Prove that M is equidistant from each of the vertices A, B and C.

Hint: Start by drawing the line through M that is parallel to the side AB.

As MN is parallel to AB

$\angle CNM = \angle CAB$ (corresponding angles)

$\angle C$ is shared so

ΔABC is similar to ΔNMC (equi-angular)

$\therefore \frac{CN}{CA} = \frac{CM}{CB} = \frac{1}{2}$ (M is midpoint of BC)

$\therefore CA = 2CN \Rightarrow$ N is midpoint of AC

$\therefore CN = NA$

As $\angle ANM = \angle CNM = 90°$ (estabished above)

$CN = NA$ and MN is shared

$\therefore \Delta CNM$ is congruent to ΔANM (SAS)

$\therefore MA = MC$ (corres. sides of congruent Δ's)

$\therefore MA = MC = MB$

$\therefore$ M is equidistant to A, B and C

Question 16 **(11 marks)**

The mean μ and standard deviation σ of the uniform distribution on the interval [a, b] are given by

$$\mu = \frac{a+b}{2} \text{ and } \sigma = \frac{b-a}{2\sqrt{3}}.$$

A calculator can generate random numbers that are uniformly distributed between 0 and 1.

(a) For this distribution of the random numbers generated by the calculator, calculate

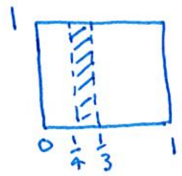
(i) the mean. (1 mark)

$\mu = \frac{1}{2}$

(ii) the standard deviation (to **three (3)** decimal places). (1 mark)

$\sigma = \frac{1}{2\sqrt{3}} \approx 0.289$

(b) What is the probability that a randomly-generated number lies between $\frac{1}{4}$ and $\frac{1}{3}$? (1 mark)



Prob = Shaded area

$= \left(\frac{1}{3} - \frac{1}{4}\right) \times 1$

$= \frac{1}{12}$

(c) What is the probability that a randomly-generated number contains no seven in its first **five (5)** decimal places? (1 mark)

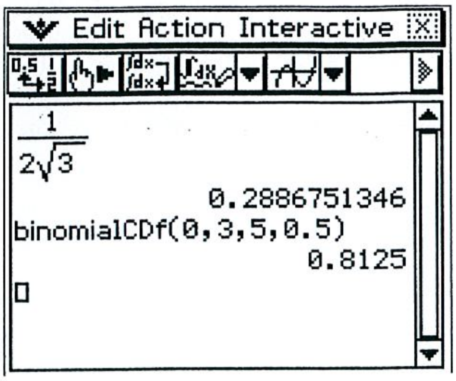
$0.9^5 = 0.59049$

(d) What is the probability that a randomly-generated number contains at most three odd digits in its first five decimal places? Give your answer to **four (4)** decimal places. (2 marks)

X = no. of odd digits in 5 places

$X \sim \text{Bin}(5, \frac{1}{2})$

$P(X \leq 3) = 0.8125$

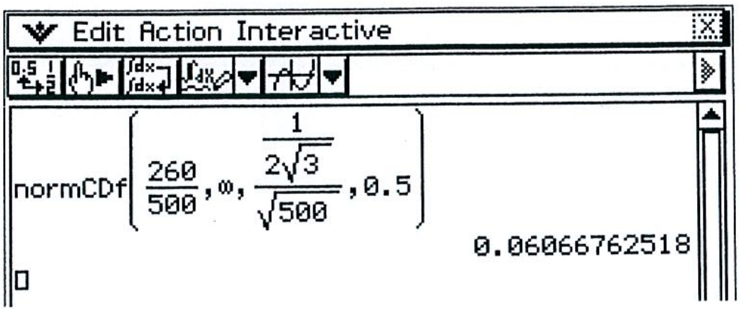


(e) What is the probability that the sum of 500 randomly-generated numbers exceeds 260? Give your answer to **four (4)** decimal places. (3 marks)

By the CLT, the mean of samples of 500 random numbers is $N\left(0.5, \frac{\frac{1}{2\sqrt{3}}}{\sqrt{500}}\right)$

Sum $> 260 \Rightarrow \bar{x} > \frac{260}{500}$

$P\left(\bar{x} > \frac{260}{500}\right) = 0.0607$



(f) Another uniform distribution on an interval $[a,b]$ has a standard deviation of $2\sqrt{3}$. How wide is the interval? (2 marks)

$\sigma = \frac{b-a}{2\sqrt{3}} = 2\sqrt{3}$

$\therefore b - a = 12$

$\therefore$ width is 12.

Question 17 **(7 marks)**

Let A denote the average of the squares of two numbers x and y:

$$A = \frac{1}{2}\left(x^2 + y^2\right),$$

and let B denote the square of the average of x and y:

$$B = \left(\frac{x+y}{2}\right)^2.$$

(a) Evaluate A and B in the case $x = 5$ and $y = 7$. (1 mark)

$A = \frac{1}{2}(25 + 49) = 37$

$B = \left(\frac{5+7}{2}\right)^2 = 36$

(b) What can be said about A and B if $x = y$? Justify your answer. (2 marks)

If $x = y$

$A = \frac{1}{2} \times 2x^2 = x^2$

$B = \left(\frac{2x}{2}\right)^2 = x$

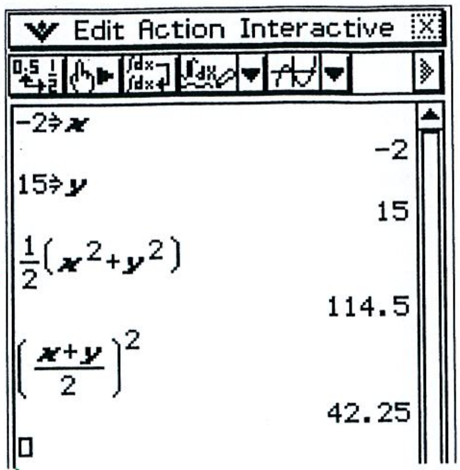
$\therefore$ $A = B$ if $x = y$

(c) Examine the difference between A and B for various values of x and y and state a conjecture about A and B. (1 mark)

$A \geq B$ for all values of x, y

Note:
CP use may not be an efficient method for students who compute well mentally.

CP Note:
Using a structure like the one shown, and varying the values assigned to x and y, A and B can be investigated.



(d) Prove the conjecture in part (c). (3 marks)

$A - B$

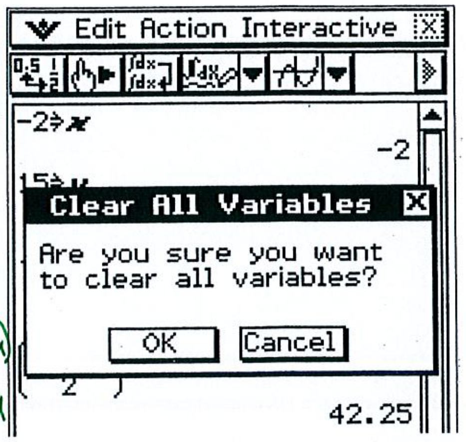
$= \frac{1}{2}(x^2+y^2) - \left(\frac{x+y}{2}\right)^2$

$= \frac{(x-y)^2}{4}$

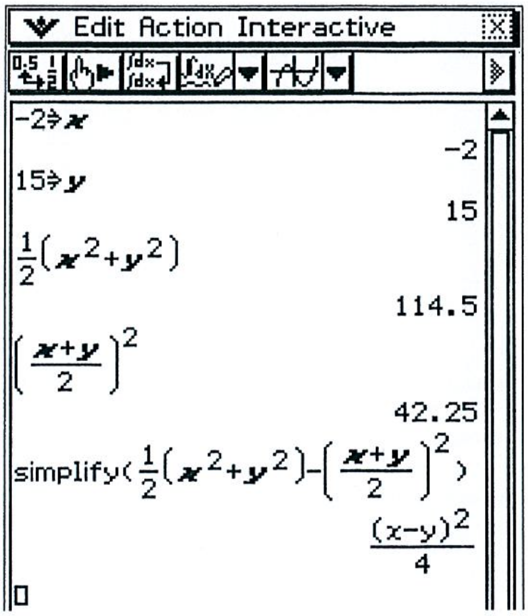
≥ 0

$\therefore A \geq B$ for all x, y

If an approach like the one above is used, "Clear all variables" is needed (via Edit menu) before x and y can be used as variables.



Note:
CP use may not be an efficient method for students who work well algebraically

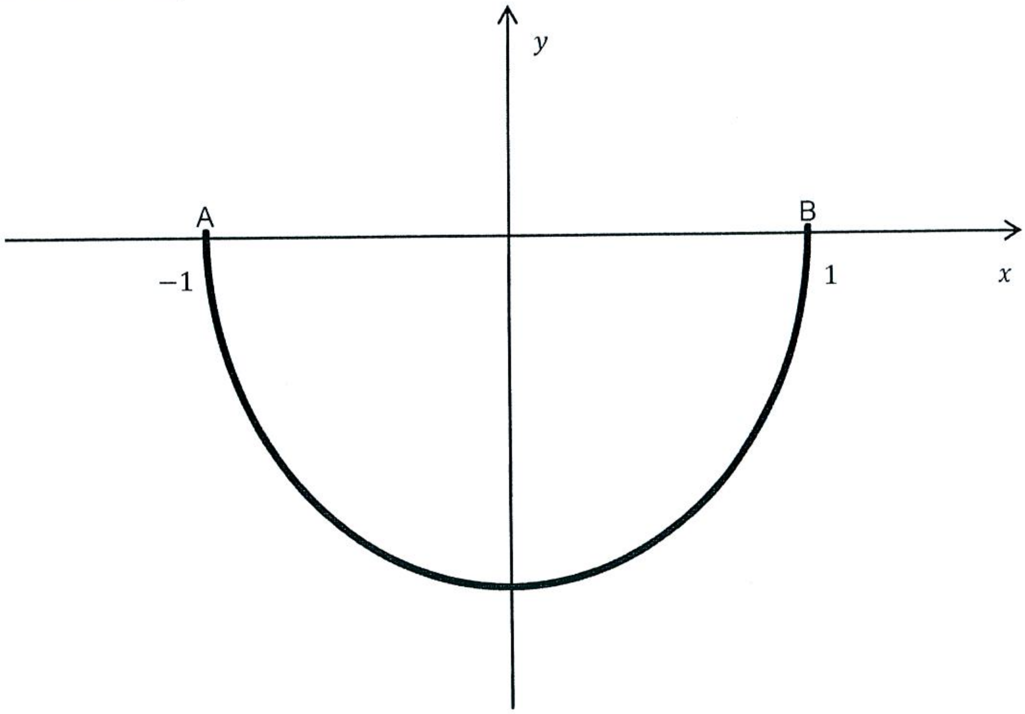


Question 18 **(7 marks)**

A cable hanging between two points $A(-1,0)$ and $B(1,0)$ lies on the curve

$$y = e^{cx} - d + e^{-cx},$$

where c and d are positive constants.



(a) Show that $d = e^{c} + e^{-c}$. (1 mark)

at $(1,0)$

$0 = e^{c} - d + e^{-c}$

$\therefore\ d = e^{c} + e^{-c}$

(b) Use calculus to show that the lowest point on the cable occurs where it crosses the y-axis, that is, where $x = 0$. (3 marks)

$$\frac{dy}{dx} = ce^{cx} - ce^{-cx}$$

$\frac{dy}{dx} = 0$ when

$$e^{cx} = e^{-cx}$$

$\therefore \quad cx = -cx$

$\therefore \quad x = 0$

Edit Action Interactive

$$\frac{d}{dx}\left(e^{c\cdot x}-e^{c}-e^{-c}+e^{-c\cdot x}\right)$$

$$\left(c\cdot e^{2\cdot c\cdot x}-c\right)\cdot e^{-c\cdot x}$$

$$\frac{dy}{dx} = (ce^{2cx} - c)e^{-cx}$$

$\frac{dy}{dx} = 0$ when $ce^{2cx} - c = 0$

$\therefore \quad e^{2cx} = 1$

$\therefore \quad x = 0$

(c) The length s of the curve $y = f(x)$, between the limits $x = a$ and $x = b$, is given by the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Use this formula to determine the length of the cable if the lowest point of the cable is 10 units below the level of the supports A and B. (3 marks)

when $x = 0$

$$e^{cx} - e^{c} - e^{-c} + e^{-cx} = -10$$

$\therefore \quad c = \ln(\sqrt{35} + 6)$

For this value of c

$$s = \int_{-1}^{1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$= 20.271$ m

Edit Action Interactive

$$\frac{d}{dx}\left(e^{c\cdot x}-e^{c}-e^{-c}+e^{-c\cdot x}\right)$$

$$\left(c\cdot e^{2\cdot c\cdot x}-c\right)\cdot e^{-c\cdot x}$$

$$\text{solve}\left(e^{c\cdot x}-e^{c}-e^{-c}+e^{-c\cdot x}=-10 \mid x=0, c\right)$$

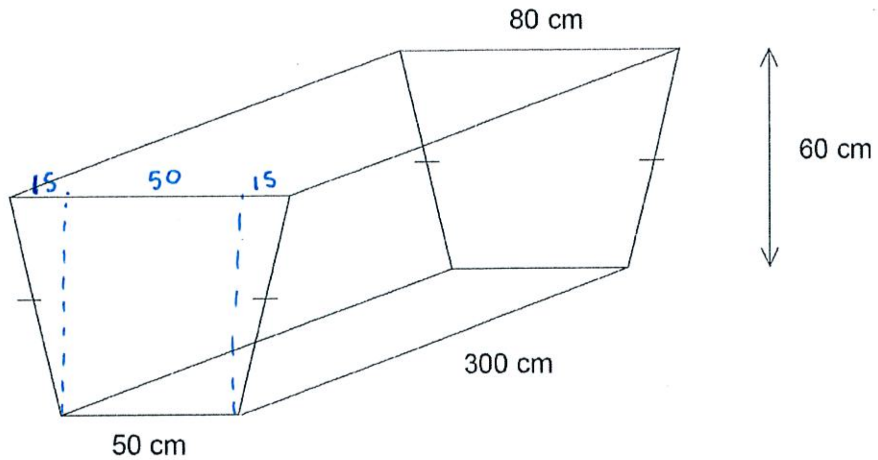
$$\left\{c=\ln\left(-\sqrt{35}+6\right), c=\ln\left(\sqrt{35}+6\right)\right\}$$

$$\int_{-1}^{1}\sqrt{1+\left(\left(c\cdot e^{2\cdot c\cdot x}-c\right)\cdot e^{-c\cdot x} \mid c=\ln\left(\sqrt{35}+6\right)\right)^2}\,dx$$

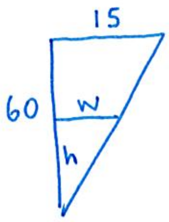
$$20.27118967$$

Question 19 **(8 marks)**

A water trough has the shape of a trapezoidal prism. It is 300 centimetres long, 60 centimetres high, 80 centimetres wide at the top and 50 centimetres wide at the bottom. A sketch of the trough is shown below (not to scale).



(a) The top surface of the water in the trough has the shape of a rectangle whose length is 300 cm. Show that if the water in the trough is h cm deep, then the width of this rectangle is $50 + 0.5h$ cm. (2 marks)



$$\frac{w}{h} = \frac{15}{60}$$

$\therefore$ $w = 0.25h$

$\therefore$ Total width $= 50 + 2w$

$= 50 + 0.5h$

(b) Show that if water in the trough is h cm deep, then the volume of water, V litres, is given by $V = h(15 + 0.075h)$. (2 marks)

$$V = \left(\frac{50 + 50 + 0.5h}{2}\right) \times h \times 300 \text{ cm}^3$$

$$= h(50 + 0.25h) \times 300 \text{ cm}^3$$

$$= \frac{h(50 + 0.25h) \times 3\cancel{0}\cancel{0}}{10\cancel{0}\cancel{0}} \text{ L}$$

$$= \frac{h(150 + 0.75h)}{10}$$

$$= h(15 + 0.075h)$$

(c) Water is being pumped into the trough at a rate of 40 litres per minute. At what rate is the depth of the water increasing at the instant when the trough is half full by volume? Give your answer correct to the nearest centimetre per minute. (4 marks)

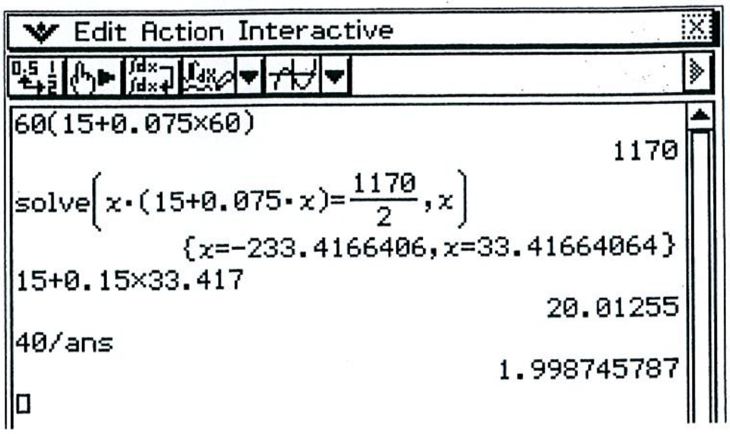
Vol. when $\frac{1}{2}$ full

$= \frac{1}{2} \times 60(15 + 0.075 \times 60)$

$= 585$ L

$V(t) = 585$ when

$t = 33.42$ mins



$\frac{dV}{dt} = \frac{dh}{dt} \times \frac{dV}{dh}$

when $t = 33.42$

$\left.\frac{dV}{dh}\right|_{t=33.42} = \left. 15 + 0.15h \right|_{t=33.42}$

$= 20.013$

$\frac{dV}{dt} = 40$ (constant rate)

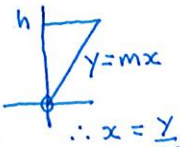
$\therefore \quad 40 = \left.\frac{dh}{dt}\right|_{t=33.42} \times 20.013$

$\left.\frac{dh}{dt}\right|_{t=33.42} = 1.9987$

$= 2$ cm/min

Question 20 **(4 marks)**

(a) Find the volume of revolution obtained when the line $y = mx$, between the limits $y = 0$ and $y = h$, is rotated about the y–axis. (2 marks)



$\therefore x = \frac{y}{m}$

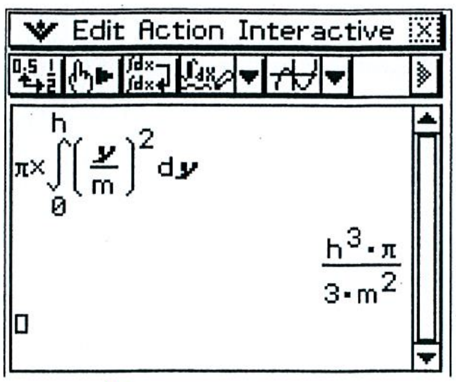
$$V = \pi \int_0^h \left[\frac{y}{m}\right]^2 dy$$

$$= \pi \int_0^h \frac{1}{m^2} y^2 dy$$

$$= \pi \left[\frac{1}{m^2} \times \frac{1}{3} y^3\right]_0^h$$

$$= \pi \times \frac{1}{m^2} \times \frac{1}{3} h^3 - 0$$

$$= \frac{\pi h^3}{3m^2}$$



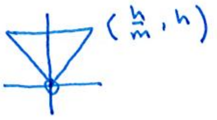
$$V = \pi \int_0^h \left[\frac{y}{m}\right]^2 dy$$

$$= \frac{\pi h^3}{3m^2}$$

(b) Use your answer from part (a) to show that the volume V of a cone is given by

$$V = \frac{1}{3} Ah$$

where A is the area of the base and h is the height. (2 marks)



radius $= \frac{h}{m}$

$\therefore$ area of base $= \pi \left(\frac{h}{m}\right)^2 = A$

From (a) $V = \frac{\pi h^3}{3m^2}$

$$= \frac{1}{3} \times \frac{\pi h^2}{m^2} \times h$$

$$= \frac{1}{3} Ah$$

End of questions

Additional working space

Question number: 12 (d) - alternative Solution

If the objective function $P = (21-n)x + (15+n)y$ has a greater slope than the edge AC then the maximum profit will shift from vertex C to vertex A. To see when this occurs consider

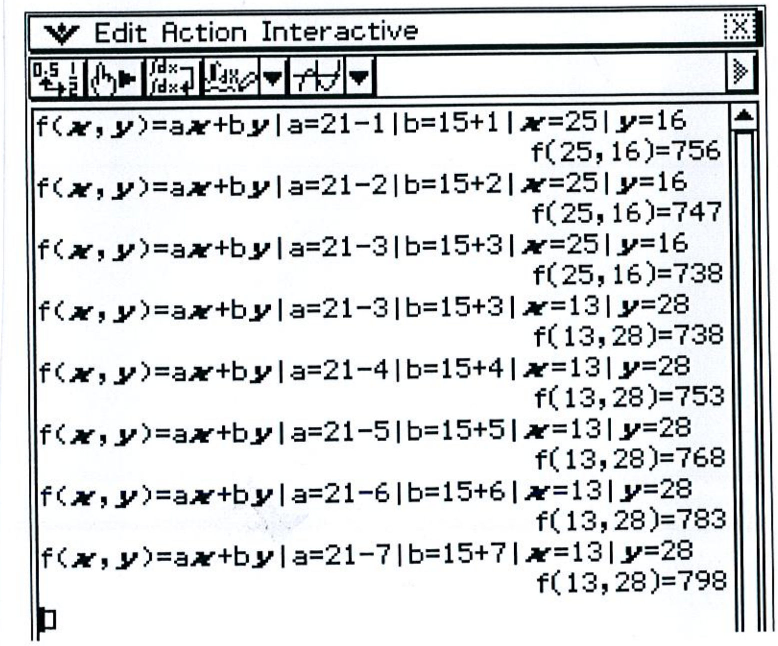
$$-\frac{21-n}{15+n} = \frac{-12}{12} = -1$$

$\therefore \quad 21 - n = 15 + n$

$\therefore \quad 2n = 6$

$\therefore \quad n = 3$

Once this is known, the maximum profit at vertex C (then A) can be found for the week, as shown →



This shows that the

maximum profit falls by \$9,000 per day for the first 3 days then the vertex changes to A and the profit rises by \$15,000 per day

End of questions