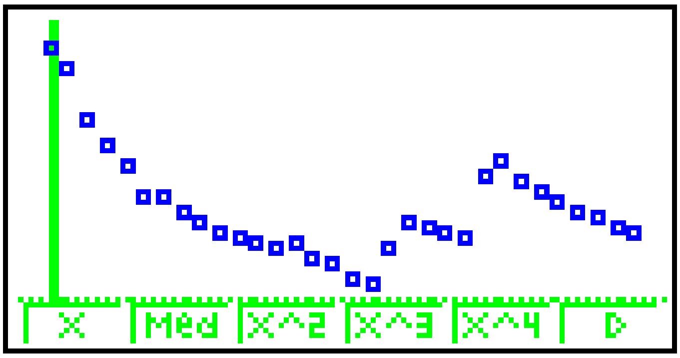
Exponential Growth

and related knowledge, skills, processes and applications.

(a non-recursive approach)



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LUMAT 2001

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CASIO

Rationale for the approach offered here.

What is offered in this booklet is a suggested approach to the introduction of what is traditionally a very difficult topic for many students.

It is not a traditional approach and is only possible if electronic technology is used.

It offers students real (non trivial) contexts to interact with, and as a consequence they learn the basic structures of the topic.

The aim is for the student to be introduced to multiplicative patterns and the notion of logarithms through examples that mean something to them.

We have not attempted to deal with the traditional parts of the course that we all know so well. Some comments are made at the end of the booklet however.

For the user.

This unit has been designed assuming you have access to some form of electronic technology at all times, including times when you are working outside the classroom. The use of electronic technology is an integral part of the learning cycle we employ.

The Casio 9850GB PLUS graphics calculator is the form of electronic technology that has been selected for use in this unit.

MS Excel or any other spreadsheet could be used if desired, but the instructions in this document refer to the Casio 9850GB PLUS Graphics Calculator.

Assumed Knowledge:

It is expected that you have a sound knowledge of:

- index laws
- the concept of a mathematical model
- using algebra to describe patterns
- basic skills in algebraic manipulation
- linear function theory, including the fitting of least squares regression lines

Learning Outcomes

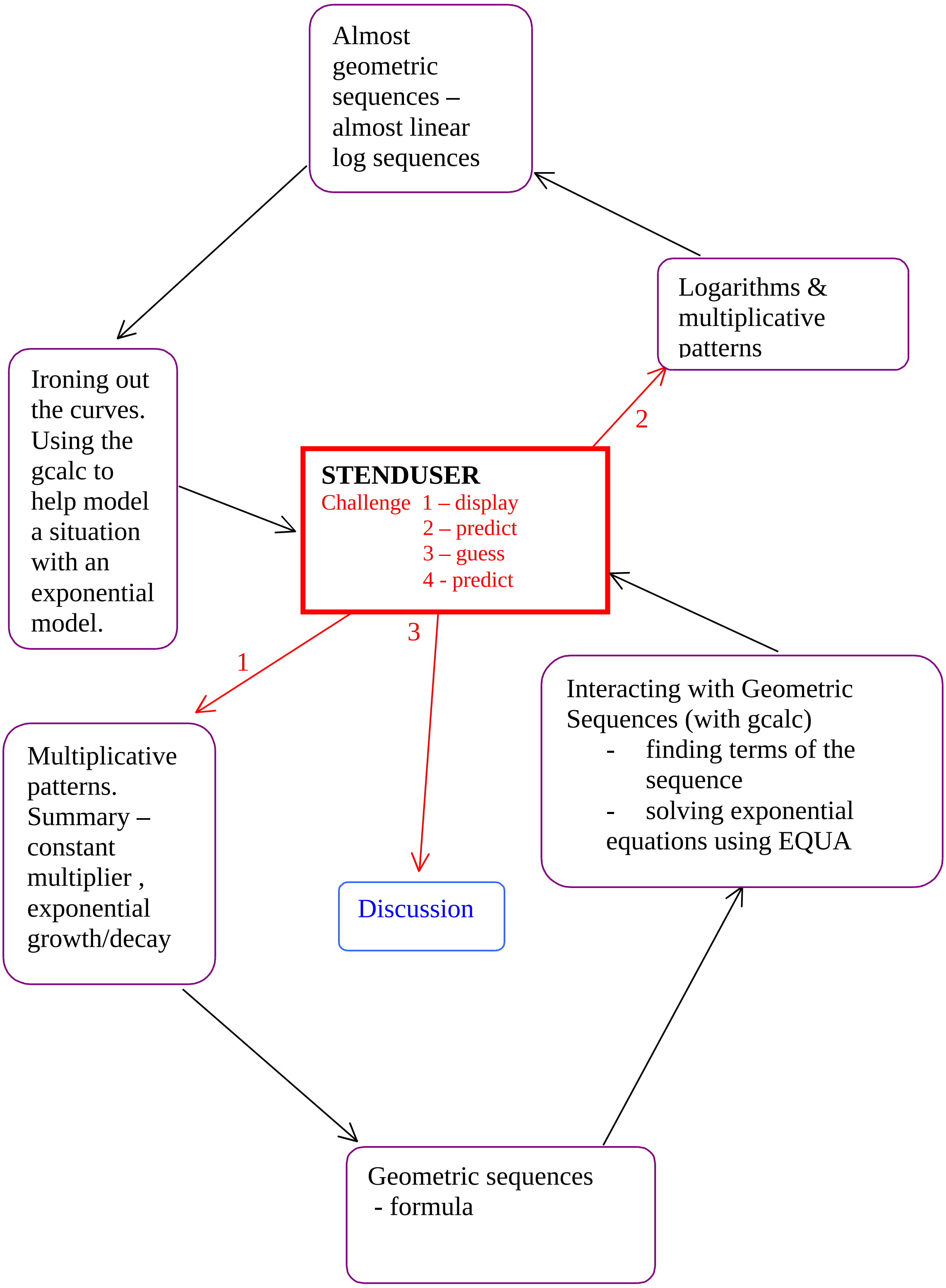
After interacting with this unit it is expected that you will:

- understand what is meant by a multiplicative pattern and the term 'constant multiplier'
- be strong users of the laws of logarithms
- develop mathematical models, using logarithms, of systems that behave in both totally predictable and highly predictable manners
- develop mathematical models, with the help of electronic technology, of systems that behave in both totally predictable and highly predictable manners
- be able to solve exponential equations both with and without the use of electronic technology
- appreciate the useful nature of solving exponential equations
- solve problems using the theory of geometric sequences and series

The Learning Cycle

The unit begins with a Stenduser. It is highly recommended that you attempt as many of the challenges that you can before doing any of the 'new' work offered in this unit. It is unlikely you will complete all of the challenges in the most accurate, appropriate, desirable or efficient manner – if at all. If you do produce a solution, discuss them with your classmates and teacher. The experience will hopefully enthuse you and give you a purpose to move on to the learning of new things. Once you have mastered the new work on offer you will be prompted to return to the Stenduser and attempt to conquer it. The flowchart on the next page may help you to visualise the learning cycle.

A flowchart of the learning cycle.



1. Stenduser: Forecasting gas production

There is a great deal of thinking and planning involved in the setting up and running of a gas production site. Geologist and Engineers work together to assess sites for the possible extraction of gas.

Once a site is chosen and a well is situated on that site the wells' productivity is monitored by measuring the flow rate of the gas out of the well.

The gas flow rate is measured every **minute**. The method of measurement is interesting in itself. A Bernoulli technique (do a search on the net and see what you find) is used. You can imagine the volume of data such frequent measuring would generate. To alleviate this problem, engineers 'coarse up' the data by calculating the average daily flow rate for a whole month. Think hard about this – describe the mathematical process used to do this. *The end result of this is that all we are interested in are the values that are set at one month apart.*

A production site may comprise a number of individual wells. The production site remains active while its productivity is viable. To be viable, the site must supply enough gas so that consumer's demands are met. There is also an issue of cost to be considered – a comparison of running costs and income from the site.

The scenario (a real one)

A gas production site in northern South Australia contains six wells. Five of the wells are installed and producing gas.

After consideration of the demand and many other factors the Reservoir Engineer in charge decides that an average daily rate for a given month must be 5 MMscf/day (millions of cubic feet per day) or greater for the site to be considered viable.

If the average daily rate for a given month falls below 5 MMscf/day the sixth well will be installed and begin to produce gas.

The table below gives the actual average daily flow rate from the site for the months shown when only five wells are installed.

Month end date	Relative time	Rate of Gas Flow (MMscf/d)
5/31/1998		51.717
6/30/1998		47.724
7/31/1998		36.717
8/31/1998		31.755
9/30/1998		28.066
10/31/1998		22.248
11/30/1998		22.199
12/31/1998		19.154
1/31/1999		16.377
2/28/1999		14.611
3/31/1999		13.403
4/30/1999		12.72
5/31/1999		11.285

Reservoir Engineers do not wait until the rate falls below the value they have set. It takes time to prepare a well for use, so they need to be able to forecast when it will be necessary to install a well.

They use the data given above to forecast (or predict) when well 6 should be installed.

Challenge One:

Complete the table on this page and produce a graphical display of the data.

Challenge Two:

Given the data, predict in what month the rate of gas flow will drop to below 5 MMscf/day and hence, when well six should be installed to boost gas production.

Challenge Three:

Use the extra data supplied below to guess when well six was installed and producing gas.

Month end date	Relative time	Rate of Gas Flow (MMscf/d)
6/30/1999		12.992
7/31/1999		9.21
8/31/1999		8.836
9/30/1999		5.874
10/31/1999		4.938
11/30/1999		11.775
12/31/1999		16.709
1/31/2000		15.579
2/29/2000		14.861
3/31/2000		14.067
4/30/2000		26.285
5/31/2000		28.882
6/30/2000		24.963
7/31/2000		23.124
8/31/2000		20.43
9/30/2000		18.963
10/31/2000		17.335
11/30/2000		15.61
12/31/2000		14.516

Challenge Four:

The site that we have been studying had only six wells. Hence, when the average daily rate of flow falls below 5 MMscf/day after the installation of well six, the site will be closed down.

It is very important for companies to be able to forecast when such an event will occur.

Use the extra data supplied in Challenge Three to predict when this site will be shut down.

2. Multiplicative Patterns

The following activities will introduce you (or remind you) of how some systems behave in a completely predictable manner. We will use algebra to describe the behaviour and to determine what will happen for cases that we cannot physically experience.

A. Paper Folding

1. Take an A-4 (or A3 if you have one) piece of paper. Mark one face with a cross to denote this to be the uppermost face. Lay it on the table (cross upwards) and fold it in half going from left to right. Be sure to crease the fold well. Open the paper so it is A-4 sized again. It has only one crease line that has formed a valley. We will call this a valley crease. Return the paper to the 'folded in half' position and fold it in half again so that the new fold is parallel to the previous one. Open the paper so it is A-4 sized again. Notice this time that it has more valley creases but also some creases that form 'mountains' - we will call these mountain creases. Your job is to continue to fold in halves and keep track of the number of valley and mountain creases. Summarise your findings in your problem book using a table similar to the one below.

number of folds (f)	1	2	3	4	5	6
number of valley creases (V)						
number of mountain creases (M)						
total number of creases (T)						

2. Describe a pattern for the number sequences for V, M and T.

3. Look at a term and its previous term. Is there a multiple involved?
   If so, what is it for V, M and T?

4. i) Conjecture a rule that links V and f
   ii) Conjecture a rule that links M and f
   iii) Conjecture a rule that links T and f
   iv) Draw graphs to illustrate each rule.

5. Use your rules to predict how many of each type of crease will be present if the paper is folded 10 times.

6. Use your rules to predict how many of each type of crease will be present if the paper is folded 20 times.

7. It is possible to prove the correct conjectures. Investigate how you may go about this.

B. Zeno's Paradox

Zeno was a philosopher from the 5th Century BC. He stated paradoxes (seemingly contradictory statements) about the schools of thought concerning magnitude. People thought of magnitude in one of two ways:

Magnitude is infinitely divisible – this means a length may be broken up into an infinite number of smaller bits.

OR

Magnitude is made of a very large number of small indivisible atomic parts.

One of Zeno's paradoxes is written below. It refers to the idea that magnitude is infinitely divisible:

> "If a straight line segment is infinitely divisible then motion is impossible, for in order to traverse the line segment it is necessary first to reach the midpoint, and to do this one must first reach the one-quarter point, and to do this one must first reach the one-eighth point, and so on, ad infinitum.
>
> Since space is infinitely divisible, we can repeat these 'requirements' forever. Thus the runner has to reach an infinite number of 'midpoints' in a finite time. This is impossible, so the runner can never reach his goal. In general, anyone who wants to move from one point to another must meet these requirements, and so motion is impossible, and what we perceive as motion is merely an illusion.
>
> It follows that the motion can never begin"

Let's investigate the mathematics of Zeno's line segment referred to in his paradox:

Consider a line segment of length ***one unit***.

1. Find and mark the halfway position along the line segment. Call this point 1.

2. Find and mark the position halfway from the beginning of the line to the point 1. Call this point 2.

3. Find and mark the position halfway from the beginning of the line to the point 2. Call this point 3.

4. Continue for as long as is sensible.

5. Complete the following table

Point number p	0	1	2	3	4
Length of line segment up to mark point p (l)	1				

6. Look at a term and its previous term. Is there a multiple involved? What is it?

7. Determine a rule that links l with p and draw a graph that illustrates the rule.

8. Predict the length of the line segment up to point 10.

9. Predict the length of the line segment up to point 20.

10. Relate what you have just experienced back to Zeno's paradox. Where does Zeno's paradox break down?

BY THE WAY
Zeno stated another paradox concerning the idea that magnitude is made up of lots of small indivisible atomic parts:
"If time is made up of indivisible atomic instants, then a moving arrow is always at rest, for at any instant the arrow is in a fixed position. Since this is true of every instant it follows that the arrow never moves."

A summary of our findings thus far.

So far we have experienced systems that produce number patterns that:

- grow with consecutive values increasing **exactly** by some common multiplier
- decay with consecutive values decreasing **exactly** by some common multiplier

Such patterns are said to grow or decay in an **exponential** manner.

It is the **constant multiplier** between consecutive terms that is the defining feature of exponential patterns.

3. Geometric Sequences

Introduction

Number patterns that have elements in between which no other numbers exist are called sequences.

The patterns produced in the paper folding activity are sequences.

Sequences in which each consecutive term grows by a constant multiplier are called GEOMETRIC SEQUENCES.

In our paper folding example the sequence that resulted from the number of valley creases was:

1,2,4,8,……………which is a geometric sequence

The sequence that resulted from the number of mountain creases was:

0,1,3,7,……………which is **not** a geometric sequence

A rule for geometric sequences

Consider the following table that depicts a geometric sequence of first term a and which grows with constant multiplier r

n	1	2	3	4	n
T_n	a	$a \times r$	$a \times r \times r$	$a \times r \times r \times r$	??

Clearly the nth term will be:

$a \times r \times r \times r \times r \times r \times r \times r \times r \times r$………with $(n\text{-}1)$ r*'s*

which could be written as

$$ar^{n-1}$$

So, if we know that a pattern behaves in a geometric (or exponential) manner we can simply find a formula that will generate any term in the sequence.

We could forecast (with total reliability) the value of any term.

Displaying Geometric Sequences on the Graphic Calculator

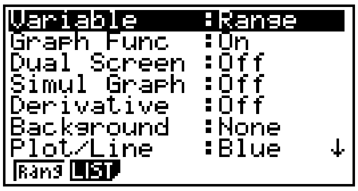
Consider the following sequence that you are told is a geometric sequence.

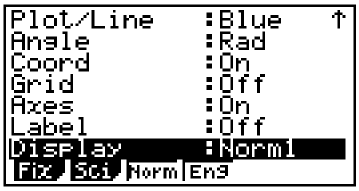
n	1	2	3	4	n
T_n	6	18	54		

The first term is 6 and the constant multiplier between consecutive terms is 3. Hence,

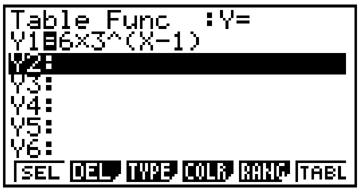
$$T_n = 6 \times 3^{n-1}$$

Enter the TABLE mode of your calculator. Use SET UP to ensure that the preferences for the table mode are set as follows:

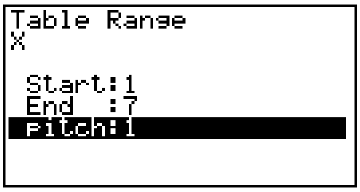




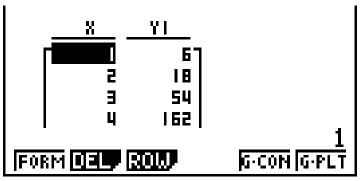
In this instance Y1 will represent T_n and x will replace n. Define Y1 as 6×3^(X-1) as seen opposite. X is entered using the X,θ,T key.



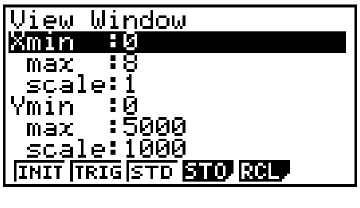
Use RANG (F5) to set the term numbers that you want to display. In this case let's view the first 7 terms, so set the parameters as shown. The pitch of 1 indicates we want the consecutive terms in the sequence.

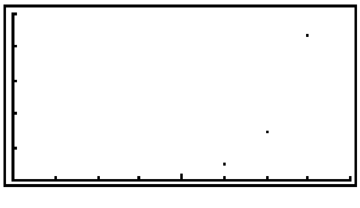


Press EXIT and then TABL (F6) to view the sequence members.



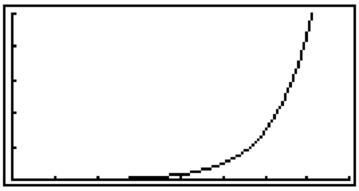
Now use V-Window (SHIFT then F3) to set the scale and end points of the axes for a graph to display the sequence as shown below. Press EXIT, produce the table again and then use G·PLT (F6) to produce the graph.





This graph illustrates to you the rapid growth of values in this sequence.

NOTE: Using G·CON (F5) to produce a graph gives a graph as seen below, which is totally inappropriate in this case.



Finding terms of a Geometric Sequence

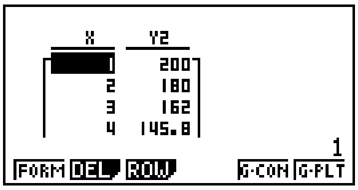
Consider the geometric sequence defined by:

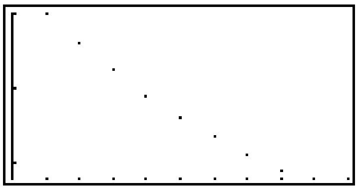
$$T_n = 200 \times (0.9)^{n-1}$$

Note that a multiplier of 0.9 indicated the sequence will be decreasing in nature, and each term will decrease by 10% from the previous (or the next will be 90% of the previous).

NB. Some practice or further exposure to the idea in italic above may be required.

Produce a table of the first 8 terms of this sequence and produce a graphical display of them. Check they are the same as ours that are seen below.



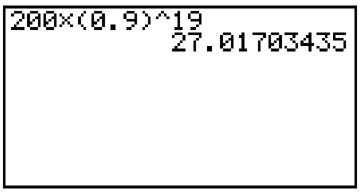


Since the graph display has no numbers on the axes you need to remember how you set the view window parameters.

If we wanted to find the 20th term in this sequence we could follow a traditional process as follows:

$$T_n = 200 \times (0.9)^{n-1}$$
$$T_{20} = 200 \times (0.9)^{20-1}$$
$$T_{20} = 200 \times (0.9)^{19}$$

and then using the RUN mode of the calculator evaluate the right hand side of the equality.

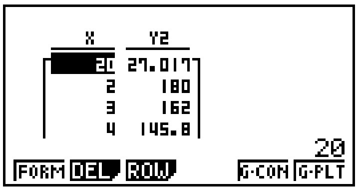


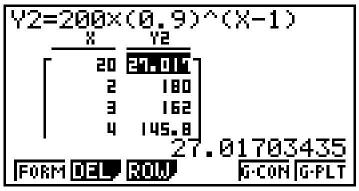
So T_{20} = 27.01703435 – I wonder what the exact answer is?

OR you could simply over type in a table that illustrates the sequence, as follows:

Produce a table of the first 8 terms again and then simply place the cursor any where in the X column. Type the term number you want, in this case 20. Hey Presto! If you use the right arrow key to put the cursor in the corresponding Y2 value, the full decimal display is seen, as is the rule for the sequence.

You could of course have simply reset the table's range.





Oh decisions, decisions, which method to use?

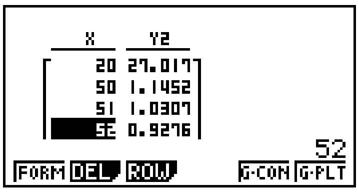
Finding the term number of a given term (solving an exponential equation) with the graphic calculator

Let us again use the geometric sequence defined by:

$$T_n = 200 \times (0.9)^{n-1}$$

Suppose we had to find the first term, which is less than 1 in this sequence.

We could simply use trial and error and over type values in the table until we reach a successful end point. The following screen show how this may be done. Each X value has been typed in.



Alternatively we could use a technique as follows:

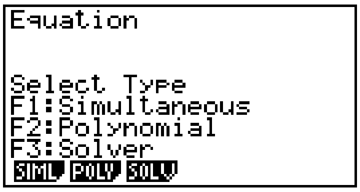
Assume that 1 is a term in the sequence, in which case we let $T_n = 1$.

This implies that:

$$1 = 200 \times (0.9)^{n-1}$$

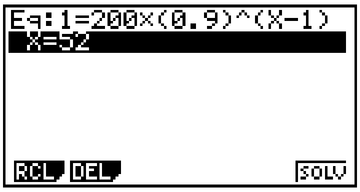
This is called an exponential equation. It can be solved (like a linear equation may be solved). The algebraic technique is a little out of our reach at this stage, but the calculator can be of assistance. We can use the SOLVER, which can solve many types of equations.

Enter the EQUA mode.

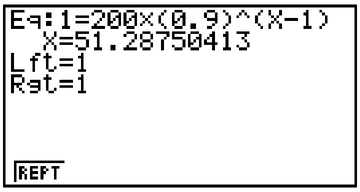


Press F3 to enter the Solver. If you have an equation in the solver already, as I do, then use DEL (F2) to delete the equation. Select YES (F1) when prompted. Or, simply highlight it and over-type it.

Enter the equation we have to solve. The = sign is accessed by pressing SHIFT and then the decimal point key. Once entered press EXE and the highlighted X=52 (or something similar) will re-appear and be highlighted. It is NOT the answer, but the value the calculator will use as its first guess as the solution. It is the last value you used as X in the calculator.



Use SOLV (F6) to find the solution. If the Lft and Rgt values are the same, an accurate solution has been found.



The solver uses Newtons-Method to solve equations. The Lft and Rgt are the values of the LHS and RHS of the equation for the value of X that Newtons Method returns.

We will develop an algebraic method of solution a little later.

Back to the Stenduser

Return to the Stenduser and see if you can apply any of the knowledge you have learned thus far in order to conquer the challenges.

Discuss your thoughts with your colleagues and teacher.

4. Logarithms – little beasties that may help to conquer the Stenduser

A. *Introduction*

The word **logarithm** is simply a fancy word for an **index**.

Consider

$$2^3 = 8$$

You would be accustomed to calling the 3 an index or exponent. It is also known as a **logarithm**. The 2 is known as the base and the 8 is called the third power of two.

Hence we can say that 'the logarithm is 3'.

Now consider

$$4^3 = 64$$

Again we could say 'the logarithm is 3'.

But this seems silly since we cannot differentiate between the two threes. Hence the following terminology was invented.

In the first case we say:

The logarithm of 8 with base 2 is 3 or $\log_2 8 = 3$

In the second case we say:

The logarithm of 64 with base 4 is 3 or $\log_4 64 = 3$

Hence we can say:

$$a^x = b \qquad \log_a b = x$$

B. *Logarithms and Multiplicative Patterns*

A simple and obvious, but powerful link exists between logarithms and geometric sequences.

Complete the following tables, given that in each case *y* is a geometric sequence, and answer the questions that follow:

x	1	2	3	4	5	6	7	8
y	8	16	32	64				
$\log_2 y$								

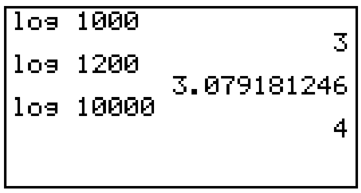
x	1	2	3	4	5	6	7	8
y	1	4	16	64				
$\log_4 y$								

x	1	2	3	4	5	6	7	8
y	10	100	1000	10000				
$\log_{10} y$								

1. What does each of the 'log sequences' have in common in the way they grow?

2. Explain why your answer to question 1 will be true for all log sequences that are generated from a geometric sequence.

3. Your calculator has the ability to calculate logarithms in base 10. In run mode you can use the log key to calculate $\log_{10}1000$. Note that the small 10 does not appear on the calculator. You can also calculate $\log_{10}1200$, $\log_{10}10000$ and so on. The logarithm in base 10 of 1200, or any number other than a power of 10 is a little tricky to think about. We will visit it later, but for now accept that the quantity exists.



Complete the following tables using your calculator and determine if your findings from the above questions hold in these situations. Pay particular attention to the last two tables. Comment after thought!

x	1	2	3	4	5	6
y	8	16	32	64	128	256
$\log_{10}y$						

x	1	2	3	4	5	6
y	200	160	128	102.4	81.92	65.536
$\log_{10}y$						

x	1	2	3	4	5	6
y	1000	98	11	1	0.11	0.01
$\log_{10}y$						

x	1	2	3	4	5	6
y	52	46	41.5	38	34.1	30.5
$\log_{10}y$						

C. *Ironing out the curves.*

You should have noted that the last two tables contained sequences that were ***almost exponential*** (or geometric) and that the ***corresponding log sequence was almost linear***.

This fact leads us to a technique for determining a rule for a sequence that is almost exponential in nature (like the one in the Stenduser).

Consider the following situation:

Scuba divers often carry tanks of compressed air on their backs while diving. They breathe this air while under the surface of the water. The compressed air contains a high percentage of Nitrogen. While under water the body of a diver experiences significantly more pressure from the water than it does from the air in our atmosphere. One effect of this is that the Nitrogen inhaled by the diver dissolves into the blood stream and is then transported into the body tissues of the diver. Upon returning to the surface and experiencing the normal pressure due to the atmosphere, the nitrogen within the tissue is expelled over a period of time.

A scientist studied the variation of the amount of Nitrogen that remains over one hour within a particular tissue type in the body once divers have returned to the surface. We will call this type of tissue - 'type A tissue'.

The table and graph below illustrates the data collected by the scientist. The amount of Nitrogen is measured as a percentage (P) remaining in type A tissue and is an average of the values gained from the divers in his study. Nitrogen readings were taken every 10 minutes.

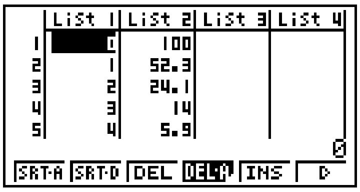
Ten minute period number	0	1	2	3	4	5	6
P (%)	100	52.3	24.1	14	5.9	3.0	1.8

Calculating consecutive ratios would give us an idea of whether or not the P values decay in an approximate exponential fashion. The ratios are approximately equal (which is equivalent to saying we have an approximately constant multiplier), suggesting an approximately exponential relationship.

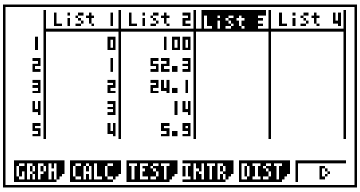
```
52.3÷100
                 0.523
24.1÷52.3
          0.4608030593
14÷24.1
          0.5809128631
```

A more thorough analysis follows in an attempt to determine an appropriate algebraic model for this data.

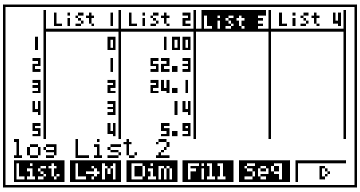
Enter the STAT mode of your calculator. In list 1 enter the period numbers and in list 2 the percentage values.



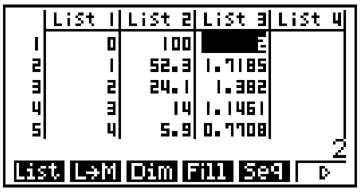
Place the cursor in the heading of List 3.



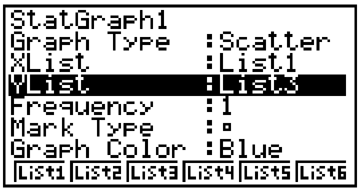
Now enter the formula `log List 2` while the cursor is in the heading of List 3. The List command is accessed by pressing OPTN (option) and then LIST (F1) and then List (F1). Finally enter 2..



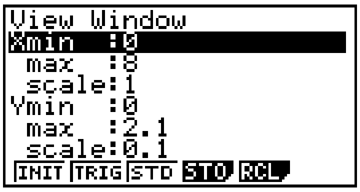
Press EXE and the logarithms with base 10 of List 2 will be calculated.



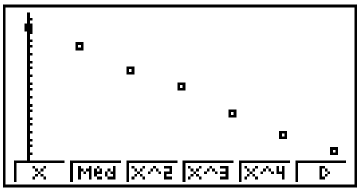
We can now investigate to see if the log sequence is approximately linear – which is done efficiently with a graph. Use GRPH (F1) and then SET (F6) to set up a graph of log P by t. Set the parameters as shown opposite.



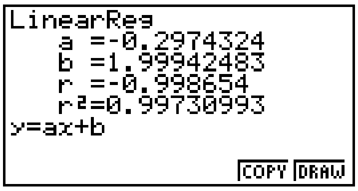
Ensure the view window is set as shown opposite, you need to consult your data to ensure it is set appropriately.

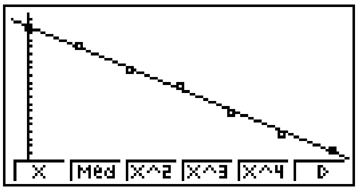


Pressing EXIT, GRPH (F1)and then GPH1(F1) gives us a plot of log P by t.



A least squares line can now be fitted to the data. Press X (F1) to achieve this. The results are displayed and then using DRAW (F6) results in the line of best fit being displayed on the graph.





Given that the graph drawn is of log P vs t, the slope and intercept values of the least squares line give us the following:

$$\mathrm{Log}_{10}\,P \approx -0.29743t + 1.9994$$

$$10^{-0.29743t + 1.9994} \approx P$$

$$10^{-0.29743t}\,10^{1.9994} \approx P$$

$$P \approx 10^{-0.29743t}\,10^{1.9994}$$

$$P \approx 100\,(0.504)^{t}$$

The 0.504 should be familiar.

Back to the Stenduser

Return to the Stenduser and see if you can apply your knowledge and conquer the challenges

Discuss your thoughts with your colleagues and teacher.

Closing comments for the teacher.

There are many questions to ponder when attacking this unit of work with students this year. We will deal with some below.

Why an exponential model?
One could fit any type of model to the 'diver' data. Try a quadratic and you may be convinced of its appropriateness for a given domain. Of course, the residuals that result from the fitting of the least squares line should be taken into account, but that is something for Stage 1 2002.

George Box is renowned as saying '**All models are wrong, but some are useful'.**

However, the whole world of theoretical modelling needs to be considered. Largely the world of differential equations, it gives us some insight into what models are appropriate for the physical situation we face. The mechanisms that control the system need to be understood if this approach is used to build a model. It is out of the scope of this course, but the students should be challenged to consider it. In the cases where an exponential model is appropriate, the system must be such that the rate of change of the quantity of interest must be able to be assumed to be proportional to the quantity.

Exponential Functions
After having used the approach suggested in this booklet it is a small jump to the notion of a continuous set of values that give rise to the exponential function. The calculator offers a quick way to graph functions of this type to see how they appear and the way they behave.

Growth and Decay Problems
The techniques offered in this booklet allow students a choice of method of solution to these traditional problems. Again a balance is required when demanding the way students approach the problems. The calculator acts as a great checking tool or as a tool to aid in the solution.

Solving exponential equations (see also the PAC policy on solving equations - attached)
The logarithmic approach to solving exponential equations will still need to be covered. It is still unclear exactly how the Year 12 examination of 2002 will change, but it is likely that it will be stipulated in some questions that algebraic solutions are required.

Logarithm rules and a proficiency in their use will also still be important to cover.

Arithmetic and Geometric Series
The calculator offers a quick way to cope with many of the problems traditionally done by students. A balance needs to be found between how students are expected to approach problems. The teaching of the use of the calculator for these tasks is not a long task.

